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Stiffness design of laminates using the polar method

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Abstract

This paper is devoted to the analysis of elastic properties of anisotropic laminas using the so-called polar representation method: this is an effective mathematical tool to analyse two-dimensional elastic problems. By this method, the authors have been able to find a particular class of solutions to some special inverse problems concerning laminates made by anisotropic layers. The properties of these solutions are described and discussed, along with some general results. © 2001 Published by Elsevier Science Ltd.

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1. Introduction

Fibre reinforced composite materials are more and more used in modern engineering, specially under the form of laminates for high performance applications. Evidently, the presence of a reinforcement in an isotropic matrix results in a general anisotropy for laminas; laminates obtained by superposing laminas with different orientations are then, in general, anisotropic as well, and their analysis by the classical laminated plate theory (CLPT) needs the transformation of elastic properties of laminas by rotation. This transformation, though well known, is rather cumbersome, because it involves the fourth powers of trigonometric functions. Naturally, this condition is an obstacle to analytical manipulations, similar to those which occur in treating inverse problems for laminates; again, the complication of formulas can hide some mechanical properties.

To overcome all this, the polar representation method of plane elasticity tensors can be used. This method was introduced by Verchery as early as 1979, and successively it has been developed by Verchery and co-workers (1986–1999) [Grédiac et al., 1993; Kandil and Verchery, 1988, 1990; Vannucci et al., 1999; Vannucci and Verchery, 1999; Verchery, 1999; Verchery and Gong, 1999; Verchery and Vong, 1986] it is

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very similar to that presented by Tsai and Pagano (1968); see also Jones (1975) and Grédiac (1996). In the theoretical framework of plane stress state, which is at the basis of CLPT, the polar method can be usefully used; its main advantage consists in the fact that material and frame rotations are easily expressed, giving the scientist much simpler equations than those obtained by Cartesian transformations.

Another important feature of the method is the physical meaning of tensor polar components: in the case, for instance, of the elastic tensor, these are invariant parameters directly representing the symmetries of a given material. So, by polar components it is possible to distinguish at a glance the kind of anisotropy of a lamina, independently on the reference frame where its elastic moduli are known.

In designing laminates, it is well known that, generally speaking, the in- and out-of-plane behaviours will be coupled, and the laminate will exhibit different elastic properties in membrane and bending for each direction. No general solution is known today to counter these effects; normally, to obtain uncoupled laminates, symmetrical stacking sequences are used, but this is only a sufficient and rather limiting rule, not a necessary one, as already shown by Caprino and Crivelli-Visconti (1982). The second effect is often neglected by designers.

In this paper, the CLPT equations are briefly recapitulated, and then written in the polar method context; we also give the rules for obtaining the tensors describing elastic properties of a laminate obtained by superposing and perfectly bonding together two different laminates. Next, we consider the possibility of finding uncoupled laminates, or laminates having the same elastic characteristics in membrane and bending, and finally the existence of laminates having both these properties, the so-called *quasi-homogeneous* laminates; these are three special inverse problems concerning the design of anisotropic laminates. The existence of a particular class of solutions common to the above mentioned inverse problems is shown; the authors have called *quasi-trivial* these solutions, to signify the fact that there is no need to directly solve the governing equations to find them. Finally, the properties of these solutions are illustrated, and some general results are shown.

2. Recall of the classical laminated plate theory

The CLPT, (see Jones, 1975; Tsai and Han, 1980; Tsai, 1985) provides the tool for the mechanical analysis of laminates. Let us consider a laminate composed of a given number, n, of anisotropic laminas, perfectly bonded together. In the hypothesis of linear elasticity, for the case of a plane state of stress, stresses and strains are related by the generalised Hooke–Duhamel law, which, in a contracted form, is

$$\boldsymbol{\sigma} = \mathbf{Q}(\boldsymbol{\varepsilon} - \tau\boldsymbol{\alpha}), \tag{1}$$

where $\mathbf{Q}$ is the plane stress stiffness tensor. The inverse of Eq. (1) is

$$\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma} + \tau\boldsymbol{\alpha}, \tag{2}$$

$\mathbf{S} = \mathbf{Q}^{-1}$ being the plane stress compliance tensor. For a linear variation of temperature across the thickness of the laminate,

$$\tau = \tau_0 + \Delta\tau z, \tag{3}$$

in-plane forces and bending moments are linked to middle-plane strains and curvatures by the classical relations

$$\begin{aligned} \mathbf{N} &= \mathbf{A}\boldsymbol{\varepsilon}^0 + \mathbf{B}\boldsymbol{\chi} - \tau_0\mathbf{U} - 2\frac{\Delta\tau}{h}\mathbf{V}, \\ \mathbf{M} &= \mathbf{B}\boldsymbol{\varepsilon}^0 + \mathbf{D}\boldsymbol{\chi} - \tau_0\mathbf{V} - 2\frac{\Delta\tau}{h}\mathbf{W}. \end{aligned} \tag{4}$$

Hereon, $\mathbf{N}$ and $\mathbf{M}$ are the tensors of in-plane forces and bending moments, $\boldsymbol{\varepsilon}^0$ is the tensor of in-plane strains for the middle plane, $\boldsymbol{\chi}$ is the tensor of curvatures, τ_0 is the difference of temperature of the middle plane with respect to a no-strain state, $\Delta\tau$ is the difference of temperature between the upper and lower face, and h is the total thickness of the plate. It must be remarked that these relations are correct only if the different layers that compose the laminate share the same thermal properties, which is the case, of course, of laminates composed of identical plies. The fourth order tensors **A** and **D** describe respectively the in- and out-of-plane behaviour of the plate, while **B** takes into account the coupling between these two behaviours. In other words, if **B** is not the null tensor, the laminate is said to be coupled, that is for an in-plane effort it will exhibit also curvatures, while for a pure bending state, the middle plane will have non-null strains. It must also be remarked that, in general, tensors **A** and **D** are not equal; this fact tells us that for membrane and bending the laminate behaves as two different homogeneous plates. The second order tensors **U**, **V** and **W** play the same role as **A**, **B** and **D** respectively, as far as the efforts produced by thermal strains are concerned. The above tensors are given by the following relations:

$$\begin{aligned}\mathbf{A} &= \sum_{k=-p}^{p} \mathbf{Q}_k(\delta_k)(z_k - z_{k-1}),\\ \mathbf{B} &= \frac{1}{2}\sum_{k=-p}^{p} \mathbf{Q}_k(\delta_k)\left(z_k^2 - z_{k-1}^2\right),\\ \mathbf{D} &= \frac{1}{3}\sum_{k=-p}^{p} \mathbf{Q}_k(\delta_k)\left(z_k^3 - z_{k-1}^3\right),\end{aligned} \tag{5}$$

where $\mathbf{Q}_k(\delta_k)$ is the stiffness tensor of the kth ply, whose material frame (the frame in which the elastic components of the material are known) forms the angle δ_k with the global reference frame of the laminate; $n = 2p$ if even, $n = 2p + 1$ if odd; z are the distances of the interfaces from the middle plane. For our convenience, we have numbered the interfaces in a somewhat unusual but effective way, see Fig. 1; in this way it is, for the case of layers of identical thickness $h_L = h/n$,

$$\begin{aligned}&z_k = \tfrac{2k+1}{2} h_L, \quad z_{k-1} = \tfrac{2k-1}{2} h_L, \quad \text{for } n = 2p+1,\\ &z_k = kh_L, \quad z_{k-1} = (k-1)h_L \quad \text{if } k > 0, \text{ and } n = 2p,\\ &z_k = (k+1)h_L, \quad z_{k-1} = kh_L \quad \text{if } k < 0, \text{ and } n = 2p.\end{aligned} \tag{6}$$

Again,

$$\begin{aligned}\mathbf{U} &= \sum_{k=-p}^{p} \beta_k(\delta_k)(z_k - z_{k-1}),\\ \mathbf{V} &= \frac{1}{2}\sum_{k=-p}^{p} \beta_k(\delta_k)\left(z_k^2 - z_{k-1}^2\right),\\ \mathbf{W} &= \frac{1}{3}\sum_{k=-p}^{p} \beta_k(\delta_k)\left(z_k^3 - z_{k-1}^3\right),\end{aligned} \tag{7}$$

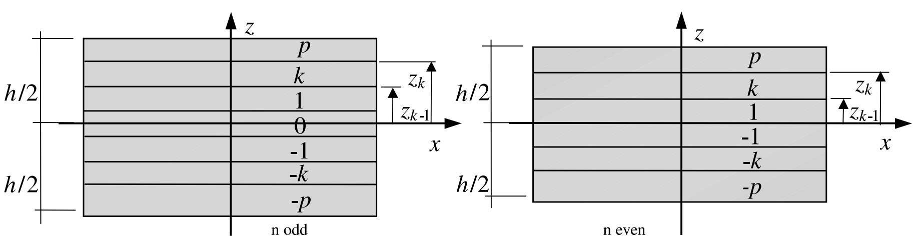

Fig. 1. General sketch for the numbering of layers and interfaces.

where $\boldsymbol{\beta}_k(\delta_k)$ is the second order tensor defined, for the kth layer, by

$$\boldsymbol{\beta}_k(\delta_k) = \mathbf{Q}_k(\delta_k)\boldsymbol{\alpha}_k(\delta_k), \tag{8}$$

$\boldsymbol{\alpha}_k(\delta_k)$ being the tensor of thermal expansion coefficients of the kth ply; of course, both $\boldsymbol{\beta}_k$ and $\boldsymbol{\alpha}_k$ must be expressed, as is $\mathbf{Q}_k$, in the global reference frame of the plate.

The above equations, which are in tensor form, are usually written in matrix form or with Cartesian components; any linear transformation of co-ordinates can be applied to them. We will show in the next paragraph how the polar method makes possible explicit formulation, including rotation angles, which has no equivalent in classical literature.

3. Superposition of laminates

Let us consider two different laminates, denoted 1 and 2; it can be easily shown that the laminate obtained by superposing these two, perfectly bonded together, has elastic properties described by the following tensors (h_1 and h_2 are the thickness of laminates 1 and 2):

$$\begin{aligned}
\mathbf{A} &= \mathbf{A}_1 + \mathbf{A}_2,\\
\mathbf{B} &= \mathbf{B}_1 + \mathbf{B}_2 - \frac{h_2}{2}\mathbf{A}_1 + \frac{h_1}{2}\mathbf{A}_2,\\
\mathbf{D} &= \mathbf{D}_1 + \mathbf{D}_2 - h_2\mathbf{B}_1 + h_1\mathbf{B}_2 + \frac{h_2^2}{4}\mathbf{A}_1 + \frac{h_1^2}{4}\mathbf{A}_2.
\end{aligned} \tag{9}$$

Similar formulae, which are a direct consequence of the composition laws of elastic tensors, Eqs. (5) and (7), can be written also for the thermo-elastic part.

4. The classical laminated plate theory using the polar method

We recall, (Verchery, 1979) that in plane elasticity, the three Cartesian components of a symmetric second order tensor $\mathbf{L}$ can be expressed by three other quantities, a scalar T, a modulus R and an angle Φ:

$$\begin{aligned}
L_{11} &= T + R\cos 2\Phi,\\
L_{22} &= T - R\cos 2\Phi,\\
L_{12} &= R\sin 2\Phi.
\end{aligned} \tag{10}$$

The reverse equations of Eq. (10) can be expressed in complex form:

$$\begin{aligned}
T &= \frac{L_{11} + L_{22}}{2},\\
2R\,\mathrm{e}^{2\mathrm{i}\Phi} &= L_{11} - L_{22} + 2\mathrm{i}L_{12}.
\end{aligned} \tag{11}$$

T, R and Φ are the polar components of $\mathbf{L}$; indeed, formulas (10) and (11) are the algebraic transposition of Mohr's circle geometric construction.

For the case of a fourth rank tensor $\mathbf{L}$ having the typical symmetries of elasticity, its six Cartesian components are a function of six other parameters, T_0, T_1, R_0, R_1, Φ_0 and Φ_1:

$$\begin{aligned} L_{1111} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 + 4R_1\cos 2\Phi_1,\\ L_{1122} &= -T_0 + 2T_1 - R_0\cos 4\Phi_0,\\ L_{2222} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 - 4R_1\cos 2\Phi_1,\\ L_{1212} &= T_0 - R_0\cos 4\Phi_0,\\ L_{1112} &= R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1,\\ L_{2212} &= -R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1. \end{aligned} \tag{12}$$

The reverse equations of Eq. (12) are, in complex form,

$$\begin{aligned} 8T_0 &= L_{1111} + L_{2222} - 2L_{1122} + 4L_{1212},\\ 8T_1 &= L_{1111} + L_{2222} + 2L_{1122},\\ 8R_0\,e^{4i\Phi_0} &= L_{1111} + L_{2222} - 2L_{1122} - 4L_{1212} + 4i(L_{1112} - L_{2212}),\\ 8R_1\,e^{2i\Phi_1} &= L_{1111} - L_{2222} + 2i(L_{1112} + L_{2212}). \end{aligned} \tag{13}$$

Parameters T_0, T_1 are scalar, R_0 and R_1 moduli, Φ_0 and Φ_1 polar angles. The most important feature of polar components, is that for a rotation θ of the reference frame, T, R, T_0, T_1, R_0, R_1 and the difference $\Phi_0 - \Phi_1$ are invariant, while the polar angles Φ, Φ_0 and Φ_1 are simply changed into $\Phi - \theta$, $\Phi_0 - \theta$ and $\Phi_1 - \theta$. This is a real advantage in the case of laminates, where laws (5) and (7) depend upon tensors expressed for each layer in a frame rotated through an angle δ_k with respect to the material frame.

The polar parameters give account of the elastic symmetries of the material composing a lamina: in fact, it appears immediately from Eqs. (10) and (12), and for what said above about a change of reference frame by the polar method, that a material will be isotropic if and only if $R = 0$, for a second rank tensor, or $R_0 = R_1 = 0$, for a fourth order tensor. So, T, T_0 and T_1 represent the isotropic part of $\mathbf{L}$, while R, R_0, R_1, Φ, Φ_0 and Φ_1 its anisotropic part. As said in the previous section, their difference $\Phi_0 - \Phi_1$ is an invariant parameter, and namely it provides the orthotropy condition: $\mathbf{L}$ is orthotropic if and only if $\Phi_0 - \Phi_1 = k\pi/4$, with k integer (Verchery, 1979, 1999).

The polar method can be effectively used to express the components of the six tensors introduced previously: a quick glance at Eqs. (11) and (13) shows that the polar components of these tensors will be found using the same laws (5) and (7) that apply for the Cartesian components. If T_{0_k}, T_{1_k}, R_{0_k}, R_{1_k}, Φ_{0_k} and Φ_{0_k} indicate the polar components of the tensor $\mathbf{Q}_k$ and T_k, R_k and Φ_k those of tensor $\boldsymbol{\beta}_k$, the results are:

tensor $\mathbf{A}$:

$$\begin{aligned} \overline{T}_0 &= \sum_{k=-p}^{p} T_{0_k}(z_k - z_{k-1}),\\ \overline{T}_1 &= \sum_{k=-p}^{p} T_{1_k}(z_k - z_{k-1}),\\ \overline{R}_0\,e^{4i\bar{\Phi}_0} &= \sum_{k=-p}^{p} R_{0_k}\,e^{4i(\Phi_{0k}+\delta_k)}(z_k - z_{k-1}),\\ \overline{R}_1\,e^{2i\bar{\Phi}_1} &= \sum_{k=-p}^{p} R_{1_k}\,e^{2i(\Phi_{1k}+\delta_k)}(z_k - z_{k-1}), \end{aligned} \tag{14}$$

tensor **B**:

$$\begin{aligned}
\widehat{T}_0 &= \frac{1}{2}\sum_{k=-p}^{p} T_{0_k}\left(z_k^2 - z_{k-1}^2\right),\\
\widehat{T}_1 &= \frac{1}{2}\sum_{k=-p}^{p} T_{1_k}\left(z_k^2 - z_{k-1}^2\right),\\
\widehat{R}_0\,\mathrm{e}^{4\mathrm{i}\hat{\Phi}_0} &= \frac{1}{2}\sum_{k=-p}^{p} R_{0_k}\,\mathrm{e}^{4\mathrm{i}(\Phi_{0k}+\delta_k)}\left(z_k^2 - z_{k-1}^2\right),\\
\widehat{R}_1\,\mathrm{e}^{2\mathrm{i}\hat{\Phi}_1} &= \frac{1}{2}\sum_{k=-p}^{p} R_{1_k}\,\mathrm{e}^{2\mathrm{i}(\Phi_{1k}+\delta_k)}\left(z_k^2 - z_{k-1}^2\right),
\end{aligned} \tag{15}$$

tensor **D**:

$$\begin{aligned}
\widetilde{T}_0 &= \frac{1}{3}\sum_{k=-p}^{p} T_{0_k}\left(z_k^3 - z_{k-1}^3\right),\\
\widetilde{T}_1 &= \frac{1}{3}\sum_{k=-p}^{p} T_{1_k}\left(z_k^3 - z_{k-1}^3\right),\\
\widetilde{R}_0\,\mathrm{e}^{4\mathrm{i}\tilde{\Phi}_0} &= \frac{1}{3}\sum_{k=-p}^{p} R_{0_k}\,\mathrm{e}^{4\mathrm{i}(\Phi_{0k}+\delta_k)}\left(z_k^3 - z_{k-1}^3\right),\\
\widetilde{R}_1\,\mathrm{e}^{2\mathrm{i}\tilde{\Phi}_1} &= \frac{1}{3}\sum_{k=-p}^{p} R_{1_k}\,\mathrm{e}^{2\mathrm{i}(\Phi_{1k}+\delta_k)}\left(z_k^3 - z_{k-1}^3\right),
\end{aligned} \tag{16}$$

tensor **U**:

$$\begin{aligned}
\overline{T} &= \sum_{k=-p}^{p} T_k\left(z_k - z_{k-1}\right),\\
\bar{R}\,\mathrm{e}^{2\mathrm{i}\bar{\Phi}} &= \sum_{k=-p}^{p} R_k\,\mathrm{e}^{2\mathrm{i}(\Phi_k+\delta_k)}\left(z_k - z_{k-1}\right),
\end{aligned} \tag{17}$$

tensor **V**:

$$\begin{aligned}
\widehat{T} &= \frac{1}{2}\sum_{k=-p}^{p} T_k\left(z_k^2 - z_{k-1}^2\right),\\
\widehat{R}\,\mathrm{e}^{2\mathrm{i}\hat{\Phi}} &= \frac{1}{2}\sum_{k=-p}^{p} R_k\,\mathrm{e}^{2\mathrm{i}(\Phi_k+\delta_k)}\left(z_k^2 - z_{k-1}^2\right),
\end{aligned} \tag{18}$$

tensor **W**:

$$\begin{aligned}
\widetilde{T} &= \frac{1}{3}\sum_{k=-p}^{p} T_k\left(z_k^3 - z_{k-1}^3\right),\\
\widetilde{R}\,\mathrm{e}^{2\mathrm{i}\tilde{\Phi}} &= \frac{1}{3}\sum_{k=-p}^{p} R_k\,\mathrm{e}^{2\mathrm{i}(\Phi_k+\delta_k)}\left(z_k^3 - z_{k-1}^3\right),
\end{aligned} \tag{19}$$

As announced above, and unlike Cartesian formulas, these laws contain explicitly the variations of the quantities with the angles δ_k.

Once the polar components of a laminate are known, it is an easy task to come back to the Cartesian ones, by Eqs. (10) and (12), not only for the reference frame, but also for each direction rotated by an angle θ with respect to the latter.

5. Formulation of three inverse problems concerning laminates

In the preceding paragraph we have shown a simple, entirely computational utilisation of the polar method in laminate analysis; however, the method finds its best application in theoretical investigations, where the analytical simplifications given by the method itself allow the treatment of more complex problems; in this sense, we draw our attention now to three connected inverse problems concerning laminates.

We have already introduced, in the preceding paragraph, two circumstances which are common in laminate designing: the first one is coupling. We try to give a general answer to the following question: which are the necessary and sufficient conditions to have uncoupled laminates? In the introduction, we have already said that normally, to have an uncoupled laminate symmetrical stacking sequences are chosen; but this is only a sufficient, and rather limiting, condition; Caprino and Crivelli-Visconti (1982) and subsequently Kandil and Verchery (1988), have shown some counter examples of uncoupled non-symmetrical laminates.

The second circumstance is the fact that thermo-elastic properties are different in membrane and bending: in other words, the laminate behaves in the two cases as if it were constituted by different materials. We look for the conditions that give laminates having the same properties for in- and out-of-plane behaviour. The interest in this kind of laminate is due to the fact that in-plane properties, see first part of Eqs. (16) and (17), do not depend upon the position of a layer in the stacking sequence, and so they are much simpler than the equations describing the bending behaviour. Then, if a laminate has the same bending behaviour as in tension, in an optimisation phase only the latter needs to be considered, which is much easier to handle in computation: bending properties will be automatically optimised.

A third case that we have considered is the possibility of having laminates that are uncoupled and that have at the same time the same properties in membrane and bending; in other words, we look for laminates having the two preceding requirements; we will call these laminates quasi-homogeneous. Kandil and Verchery had already considered this possibility, but they had called quasi-homogeneous laminates only those corresponding to the second case, with no uncoupling condition.

The above three cases correspond to three different but linked inverse problems concerning anisotropic laminates, and we want to write their governing equations. The conditions to have uncoupled laminates are simply

$$\mathbf{B} = \mathbf{0}, \quad \mathbf{V} = \mathbf{0}. \tag{20}$$

To express the conditions giving laminates with the same properties in membrane and bending, let us introduce the following tensors:

$$\begin{aligned} \mathbf{C} &= \frac{1}{h}\mathbf{A} - \frac{12}{h^3}\mathbf{D}, \\ \mathbf{Z} &= \frac{1}{h}\mathbf{U} - \frac{12}{h^3}\mathbf{V}, \end{aligned} \tag{21}$$

where $1/h$ and $12/h^3$ are the CLPT homogenising coefficients for tensors describing membrane and bending behaviour; in this way the second inverse problem is governed simply by the conditions below:

$$\mathbf{C} = \mathbf{0}, \quad \mathbf{Z} = \mathbf{0}. \tag{22}$$

It should be noted that the uncoupling condition can be posed directly upon the laminate's coupling stiffness tensors **B** and **V**; instead, for the case of laminates having equal behaviour in tension and bending, the condition must concern normalised tensors, which are not the stiffness tensors of the plate. Such normalised tensors can be regarded as describing the behaviour of an equivalent material, by which is composed a plate having the same thickness and stiffness of the laminate.

Finally, the third problem is evidently governed simultaneously by Eqs. (20) and (22). Eqs. (20) and (22) require all the components of the tensors to be null, evidently in the polar method too, and they are completely general, that is they are valid for laminates composed of any kind of layer, not necessarily identical.

A quick glance at Eq. (9) is sufficient to see that, in general, the properties $\mathbf{B} = \mathbf{0}$ and $\mathbf{C} = \mathbf{0}$ are not preserved in the superposition of two laminates. Nevertheless, let us consider two uncoupled laminates, with the same in-plane properties, that is

$$\mathbf{A}_1 = h_1\mathbf{Q}_\mathrm{L}, \qquad \mathbf{A}_2 = h_2\mathbf{Q}_\mathrm{L}, \qquad \mathbf{B}_1 = \mathbf{B}_2 = \mathbf{0}. \tag{23}$$

This is the case, for instance, of two laminates composed of the same elementary layer, and which have stacking sequences able to respect conditions (23); $\mathbf{Q}_\mathrm{L}$ is the homogenised tension stiffness tensor. It is evident from Eq. (9), that in such a case the laminate obtained by superposition will have

$$\mathbf{A} = h\mathbf{Q}_\mathrm{L}, \qquad \mathbf{B} = \mathbf{0}, \tag{24}$$

where h is the thickness of the final laminate, $h = h_1 + h_2$; Eq. (24) shows that the laminate obtained by superposition has the same equivalent properties of its two constituent laminates.

In addition, if these laminates are also quasi-homogeneous, the resulting laminate will be so; this can be easily shown if we consider that quasi-homogeneity implies

$$\begin{aligned} \mathbf{D}_1 &= \frac{h_1^2}{12}\mathbf{A}_1 = \frac{h_1^3}{12}\mathbf{Q}_\mathrm{L}, \\ \mathbf{D}_2 &= \frac{h_2^2}{12}\mathbf{A}_2 = \frac{h_2^3}{12}\mathbf{Q}_\mathrm{L}. \end{aligned} \tag{25}$$

Substituting Eqs. (23) and (25) into the third part of Eq. (9) gives directly

$$\mathbf{D} = \frac{h^3}{12}\mathbf{Q}_\mathrm{L}, \tag{26}$$

which confirms what was stated above: the laminate obtained by superposition is quasi-homogeneous.

Clearly, what is true for the superposition of two laminates can be easily extended to the case of the superposition of more than two laminates.

6. Laminates composed by identical layers

In order to have general solutions to the three problems mentioned, we make now the hypothesis that all the plies of the laminate be identical, that is composed by the same material and of the same thickness $h_\mathrm{L} = h/n$.

This assumption leads to a general simplification of Eqs. (14)–(19), because polar components of the plies are the same, say T_0, T_1, R_0, R_1, Φ_0, Φ_1, T, R and Φ; one can immediately see that in this case it is,

$$\widehat{T}_0 = \widehat{T}_1 = \widehat{T} = 0, \tag{27}$$

and, if we consider membrane and bending stiffness tensors, homogenised as in Eq. (21),

$$\begin{aligned}\overline{T}_0 &= \widetilde{T}_0 = T_0,\\ \overline{T}_1 &= \widetilde{T}_1 = T_1,\\ \overline{T} &= \widetilde{T} = T.\end{aligned} \tag{28}$$

Eqs. (27) and (28) show that the isotropic part of the stiffness tensors as well as the spherical part of thermo-elastic tensors, automatically satisfy the governing equations for the three problems; so, only the anisotropic, for stiffness tensors, and deviatoric part, for thermo-elastic tensors, play a role in the defined problems; this is a consequence of having assumed laminates made of identical plies. In addition, homogenised T_0, T_1 and T components for membrane and bending of the laminate are identical to their counterparts of the single ply.

Relations (27) and (28) are completely general, valid for each kind of material composing the elementary layer; in addition, it can be easily shown, using Eqs. (14)–(16), that for cross-ply laminates it is also

$$\begin{aligned}\bar{R}_0 &= \widetilde{R}_0 = R_0,\\ \bar{\Phi}_0 &= \tilde{\Phi}_0 = \Phi_0,\\ \widehat{R}_0 &= 0.\end{aligned} \tag{29}$$

It is worth noting that similar relations do not hold for the thermo-elastic part, because in this case $\bar{R}$, $\widehat{R}$ and $\widetilde{R}$ depend upon quantities which vary as 2θ, just like $\bar{R}_1$, $\widehat{R}_1$ and $\widetilde{R}_1$, and not as 4θ, like $\bar{R}_0$, $\widehat{R}_0$ and $\widetilde{R}_0$.

Again from Eq. (14) to Eq. (19), substituted in Eqs. (20) and (22), we have the governing equations of the three inverse problems; a laminate will be uncoupled if and only if

$$\begin{aligned}\widehat{R}_0\,e^{4i\hat{\Phi}_0} = 0 &\Rightarrow \sum_{k=-p}^{p} e^{4i\delta_k}\left(z_k^2 - z_{k-1}^2\right) = 0,\\ \widehat{R}_1\,e^{2i\hat{\Phi}_1} = 0 &\Rightarrow \sum_{k=-p}^{p} e^{2i\delta_k}\left(z_k^2 - z_{k-1}^2\right) = 0,\end{aligned} \tag{30}$$

for the elastic part, and

$$\widehat{R}\,e^{2i\hat{\Phi}} = 0 \Rightarrow \sum_{k=-p}^{p} e^{2i\delta_k}\left(z_k^2 - z_{k-1}^2\right) = 0, \tag{31}$$

for the thermo-elastic part. A laminate will have the same properties for in- and out-of-plane behaviour if and only if

$$\begin{aligned}\frac{1}{h}\bar{R}_0\,e^{4i\bar{\Phi}_0} = \frac{12}{h^3}\widetilde{R}_0\,e^{4i\tilde{\Phi}_0} &\Rightarrow \frac{1}{n}\sum_{k=-p}^{p} e^{4i\delta_k} = \frac{4}{h^3}\sum_{k=-p}^{p} e^{4i\delta_k}\left(z_k^3 - z_{k-1}^3\right),\\ \frac{1}{h}\overline{R}_1\,e^{2i\bar{\Phi}_1} = \frac{12}{h^3}\widetilde{R}_1\,e^{2i\tilde{\Phi}_1} &\Rightarrow \frac{1}{n}\sum_{k=-p}^{p} e^{2i\delta_k} = \frac{4}{h^3}\sum_{k=-p}^{p} e^{2i\delta_k}\left(z_k^3 - z_{k-1}^3\right),\end{aligned} \tag{32}$$

for the elastic part, and

$$\frac{1}{h}\overline{R}\,e^{2i\bar{\Phi}} = \frac{12}{h^3}\widetilde{R}\,e^{2i\tilde{\Phi}} \Rightarrow \frac{1}{n}\sum_{k=-p}^{p} e^{2i\delta_k} = \frac{4}{h^3}\sum_{k=-p}^{p} e^{2i\delta_k}\left(z_k^3 - z_{k-1}^3\right), \tag{33}$$

for the thermo-elastic one. Finally, a laminate will be quasi-homogeneous if and only if Eqs. (30)–(33) hold at the same time.

The above equations show that the thermal problems are a sub-part of the elastic problem: a laminate which is a solution of one of the problems for the elastic part will be a solution also for the thermo-elastic one; the contrary is not true, in general; so, we will consider only the elastic part, that is, Eqs. (30) and (32), and we will ignore Eqs. (31) and (33). By simple manipulations, Eqs. (30) and (32) can be rewritten as:

$$\begin{aligned}
&\sum_{k=1}^{p} b_k(\mathrm{e}^{4\mathrm{i}\delta_k} - \mathrm{e}^{4\mathrm{i}\delta_{-k}}) = 0,\\
&\sum_{k=1}^{p} b_k(\mathrm{e}^{2\mathrm{i}\delta_k} - \mathrm{e}^{2\mathrm{i}\delta_{-k}}) = 0,\\
&\sum_{k=1}^{p} c_k(\mathrm{e}^{4\mathrm{i}\delta_k} + \mathrm{e}^{4\mathrm{i}\delta_{-k}}) + c_0\,\mathrm{e}^{4\mathrm{i}\delta_0} = 0,\\
&\sum_{k=1}^{p} c_k(\mathrm{e}^{2\mathrm{i}\delta_k} + \mathrm{e}^{2\mathrm{i}\delta_{-k}}) + c_0\,\mathrm{e}^{2\mathrm{i}\delta_0} = 0.
\end{aligned} \tag{34}$$

In Eq. (34), coefficients b_k and c_k are defined by

$$\begin{aligned}
&c_k = p(p+1) - 3k^2, \qquad b_k = k \quad \text{for } n = 2p+1,\\
&c_k = (p-1)(p+1) - 3k(k-1), \qquad c_0 = 0, \qquad b_k = 2k-1 \quad \text{for } n = 2p.
\end{aligned} \tag{35}$$

Clearly, first and second part of Eq. (34) account for the problem of uncoupled laminates, third and fourth part of Eq. (34) the problem of laminates with same properties in membrane and bending, and all together the problem of quasi-homogeneous laminates. Some remarks can be made: firstly, the structure of the equations: there are two groups of coefficients and two kinds of equations: one which depends upon $2\delta_k$, and the other upon $4\delta_k$, being δ_k, the orientations of the plies with respect to the global reference frame of the laminate, the true variables of the problems. Secondly, the equations do not depend upon the elastic properties of the plies: this is another consequence of the hypothesis of identical plies, and it was to be expected. Thirdly, the trivial solution, that is a solution were all the layers have the same orientation, is really a solution of the equations, as can be quickly verified. Finally, and most important, it must be noted that a general solution of Eq. (34), that can be easily put in real form, is at present not available, and for a given number of plies, a solution must be sought by a numerical method. Naturally, this leaves some unresolved questions, such as the number of different solutions for a given laminate.

7. Quasi-trivial solutions

Numerical methods can resolve, at least approximately, the problem of finding a solution for a given number of plies, but they do not constitute a general method; in addition, they do not provide an exact solution, if any exists. So, if exact solutions and a general method are looked for, numerical methods cannot be used.

Fortunately, Eq. (34) have a characteristic structure which suggests the existence of particular solutions rather easy to be found. In fact, if we look at coefficients b_k and c_k, Eq. (35), we discover that the former are anti-symmetric and linearly variable along the thickness of the plate (that is why each symmetric stacking sequence is automatically uncoupled), while the latter have a symmetric and quadratic variation. But, and most importantly, the sum of coefficients b_k, as well as the sum of coefficients c_k, is null. So, a sufficient condition to have a solution is that the stacking sequence be composed of groups of layers with the same orientation, in such a way that the sum of the coefficients for each group is null. If it is the sum of coefficients b_k that is null, the laminate is uncoupled, if it is the sum of coefficients c_k, it has the same properties for membrane and bending, and if both the sums are null, the laminate is quasi-homogeneous. We have called a group of layers having the same orientation and null sum a *saturated group*, and *quasi-trivial* a solution composed of saturated groups, to indicate the fact that the governing equations need not to be solved directly to obtain it. In fact, quasi-trivial solutions are found by arithmetical combinations of the coefficients. It must be pointed out that a group is saturated independently from the orientation of the plies, which is not determined; this constitutes the most interesting properties of quasi-trivial solutions from a mechanical point of view, because for the same quasi-trivial solution, there are infinite different laminates, with elastic properties determined by the orientation of the groups; these orientations can be freely fixed by the designer to obtain specific elastic characteristic, always having a laminate with the property of being uncoupled, or with the same behaviour in membrane and bending or quasi-homogeneous.

The authors have used the preceding arguments to make an automatic algorithm able to find all quasi-trivial solutions for a given laminate, and for each one of the preceding inverse problems. In this algorithm, other properties of quasi-trivial solutions have been used, and namely the fact that each quasi-trivial solution with g different orientations descends from another with $g-1$ groups. In other words, let us suppose that we dispose of a solution with g groups; this means that the gth group, which is saturated, can be included in each one of the remaining $g-1$, always obtaining a solution, evidently with $g-1$ different orientations. To do this, it is sufficient to orient the gth group as another one, operation always possible thanks to the indeterminacy of the orientation of each group. If all the possible solutions with $g-1$ orientations have then been found, to find all the solutions with g groups it is sufficient to look for saturated subgroups in each one of the $g-1$ different groups. As a consequence of this property, if a laminate has not quasi-trivial solutions with g different orientations, it will not have quasi-trivial solutions also for more than g orientations; this circumstance gives a criterion for stopping the search procedure for quasi-trivial solutions, but at the same time it does not permit an a priori prediction of the number of quasi-trivial solutions for a given laminate.

It can be shown that the higher number of different saturated groups is equal to $[(n+1)/2]$ for the case of uncoupled laminates (the symbol [ ] denotes here the integer part), it is $[n/2]+1$ for the case of laminates having the same properties in membrane and bending, and it is less than $[n/2]$ for quasi-homogeneous laminates.

In the preceding paragraph we have recalled that a solution for thermo-elastic properties is not always a solution for elastic properties. Nonetheless, the set of quasi-trivial solutions for the elastic case coincides perfectly with the set of quasi-trivial solutions for the thermo-elastic case. This is due to the fact that in Eq. (34) there are only two different groups of coefficients, b_k and c_k, that appear both in linear and quadratic equations, and to the same nature of quasi-trivial solutions, which are solutions where simply one looks for groups of layers having a null sum of these coefficients.

All quasi-trivial solutions are general solutions, in the sense that they do not depend upon the kind of material composing the elementary layer. One can ponder if additional properties of the latter can induce a greater number of quasi-trivial solutions. In the case of layers having square symmetry of elastic properties, the answer is no. In fact, in this case for each layer the condition of square symmetry is given by (Verchery, 1999)

$$R_1 = 0, \tag{36}$$

which transforms the second and fourth part of Eq. (34) into identities. Nevertheless, the remaining, the first and third part of Eq. (34) again contain coefficients b_k and c_k, and so the search of saturated group does not change, that is the number of solutions does not increase, for each one of the three problems considered. We note that condition (36) gives immediately that a laminate composed of any kind of layers, but all having square symmetric elastic properties, will be square symmetric for each behaviour, that is, it will have (Vincenti et al., 2001)

$$\overline{R}_1 = \widehat{R}_1 = \widetilde{R}_1 = 0, \tag{37}$$

once more the polar method shows its effectiveness in the analysis of plane elastic properties.

After considering the great number of quasi-trivial solutions, see the following paragraph, the question arises if these are the only possible solutions; the answer is no. In fact, in order to find some non-quasi-trivial solutions, let us pose the following rather natural question: is the superposition of two quasi-trivial laminates, with respect to a given property, still a quasi-trivial laminate having the same property? Again, the answer is no, in general: indeed, conditions (23) are valid, for two given stacking sequences, only for special orientations of the saturated groups, and for a superposition of the two laminates which corresponds to that which superposes exactly the reference frames of the two composing laminates. In other words, the superposed solution is not quasi-trivial, because it depends upon particular and fixed orientations of the layers. If the two composing laminates are also isotropic, the angle of superposition has no influence; this shows the existence of another type of solution, that is, laminates composed by subgroups with fixed orientations, but with a degree of freedom which is the angle of orientation of one group with respect to another. Of course, other types of solutions can exist, for instance solutions where there is a functional relation between layer orientations: Vincenti et al. (2001) have shown the existence of some non-quasi-trivial solutions for laminates composed of identical layers having an elastic square symmetry, namely for the case of a 4- and 5-layer uncoupled and for a 6-layer quasi-homogeneous laminate. In such cases, a functional dependence of orientations has been found. This circumstance shows the existence of solutions depending on particular symmetries of the basic layer, but, as said above, these ones cannot be of quasi-trivial type. All these considerations are sufficient to show the non-uniqueness of quasi-trivial solutions.

A final remark is reserved for the search algorithm of quasi-trivial solutions: for what said above, it is apparent that this algorithm executes only a search of null sums of coefficients; the computing phase concerns then only integer numbers and is very quick and efficient; this algorithm provides all the quasi-trivial solutions for a laminate with n layers. In the following paragraph, we present some general case results.

8. General results and examples

We have examined a certain number of different cases, and using the above mentioned algorithm we have constituted a data base of quasi-trivial solutions. Some examples of these are shown below (the number of layers of the same saturated group is indicated in brackets):

7-layer uncoupled laminate: [*a b b c a a b*] (3/3/1),

8-layer laminate having identical bending and tension behaviour: [*a a b a b a b b*] (4/4),

8-layer quasi-homogeneous laminate: [*a b b a b a a b*] (4/4),

20-layer quasi-homogeneous laminate: [*a b c c d b d b c d a b c a c a b b c d*] (4/6/6/4).

Letters *a*, *b*, *c*, *d* are labels that denote layers belonging to the same saturated group, in the stacking sequence; the orientation of each group is not fixed, being up to the designer.

In Table 1, we show the results of our search; in numbering the solutions, we have not taken into account for those that can be considered as mechanically or mathematically identical. In fact, let us consider two solutions like [*a b c c b a*] and [*a c b b c a*], or two others like [*a b b a c*] and [*c a b b a*]; clearly, they are mechanically the same solution (the second case is simply the first upside down). Again, a solution like [*a b b b a*] can be considered as derived from [*a b c c b a*], simply giving to the saturated group labelled *c*, the same orientation as that labelled *b* (this operation is always possible, having quasi-trivial solutions no predetermined orientations). In this last case, the two solutions are mechanically different but mathematically

Table 1
Number of independent quasi-trivial solutions for the three inverse problems

	Number of plies														
	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\mathbf{B}=\mathbf{0}$	1	1	1	2	2	4	5	11	15	41	57	174	275	1033	1639
$\mathbf{C}=\mathbf{0}$	2	1	4	1	8	2	22	14	34	36	119	52	76	445	617
$\mathbf{B}=\mathbf{C}=\mathbf{0}$	0	0	0	1 (1)	1	0	0	3 (2)	1	4	2 (1)	4	8 (1)	23	5

the same. We call *independent* a solution which is mechanically and mathematically distinct; one can easily find mechanically distinct solutions simply numbering by a growing index the saturated groups (that is, the first in the sequence by a, the second by b and so on), while one solution is mathematically distinct if no quasi-trivial solution with a greater number of saturated groups derives from it. Only independent solutions are listed in Table 1, though the algorithm finds all the possible solutions, whose number is much higher, and then it selects the independent ones.

In brackets, we have reported the number of symmetric solutions; in the case of uncoupled laminates ($\mathbf{B}=\mathbf{0}$), there is always one and only one independent symmetric solution: it is the solution obtained by changing layer orientation at each symmetric pair of layers. In the case of laminates having the same properties in membrane and bending, ($\mathbf{C}=\mathbf{0}$), there are no symmetric solutions and also in the last case of quasi-homogeneous solutions, there are only a few symmetric solutions. This rather surprising fact shows that not only non-symmetric solutions exist, but that they are in greater number than the symmetric ones, which indeed constitutes an exception and not the rule, as generally thought. This fact highlights the rather limited panorama of symmetric solutions.

9. Conclusions

The capacity of the polar method to give rapid and synthetic indications about symmetries of elastic properties for a given material, still remains when passing to laminates, with all the advantages in formulating the equations of the CLPT: elastic properties of the laminate expressed by formulas where the dependence by the orientation of the layers appears explicitly, governing equations of the three considered inverse problems written in a simple manner, along with the possibility of easily finding a particular class of solutions, that we have called quasi-trivial, to these problems. In addition, we have shown, thanks to composition rules for superposed laminates, up till now unknown in the literature, that quasi-trivial solutions are not the only possible ones. The very important characteristic of quasi-trivial solutions, that is to have a-priori undetermined orientations of the plies, opens the way to possible optimisation procedures on the set of the solutions.

This paper has presented an application of the polar method to some inverse problems concerning laminates. Clearly, this method has other applications: in fact, the polar method is purely a mathematical technique, not a mechanical model, and it can be employed whenever phenomenon is described by two-dimensional tensors. We have presently in progress applications to various problems in the field of composites, such as higher order plate theories, plate buckling, plate vibrations.

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